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# Hamiltonian cycle in graphs $\sigma_4 \geq 2n$

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## ABSTRACT

Given a simple undirected graph $G$ with $n$ vertices, we denote by $\sigma_k$ the minimum value of the degree sum of any $k$ pairwise nonadjacent vertices. The graph $G$ is said to be hamiltonian if it contains a hamiltonian cycle (a cycle passing all vertices of $G$). The problem $HC$ (Hamiltonian Cycle) is well-known a $NPC$-problem. A lot of authors have been studied Hamiltonian Cycles in graphs with large degree sums $\sigma_k$, but only for $k = 1, 2, 3$. In this paper, we study the structure of nonhamiltonian graphs satisfying $\sigma_4 \geq 2n$, and we prove that the problem $HC$ for the graphs satisfying $\sigma_4 \geq 2nt$ is $NPC$ for $t < 1$ and is $P$ for $t \geq 1$.

## Keywords

hamiltonian cycle, $NPC$, $\sigma_4$.

## 1. INTRODUCTION

In this paper, we use definitions and notations in [4] with exception for $K_n$ the complete graph on $n$ vertices. We consider only simple undirected graphs. Given a graph $G = (V, E)$ on $n$ vertices with the vertex set $V$ and the edge set $E$. A set $A \subseteq V(G)$ is *independent* if no two of its elements are adjacent. The *independent number* of $G$, denoted by $\alpha(G)$, is defined by setting $\alpha(G) = max\{|I|: I \subseteq V(G) \text{ is independent}\}$. We use $\omega(G)$ to denote the number of connected components of $G$. The graph $G$ is *tough* (or *1-tough*) if $\omega(G - S) \leq |S|$ for every nonempty subset $S \subset V(G)$.

For two disjoint graphs $G_1$ and $G_2$, we denote by $G_1 * G_2$ the graph with the vertex set $V(G_1) \cup V(G_2)$ and the edge set $E(G_1) \cup E(G_2) \cup \{xy \mid x \in V(G_1),\ y \in V(G_2)\}$. For example, $K_2 * K_3 = K_5$. For a positive integer $k \leq \alpha$, we define $\sigma_k(G) = min\{\sum_{i=1}^{k} d(x_i): \{x_1, x_2, \dots, x_k\} \text{ is independent}\}$. In the case $k > \alpha$, set $\sigma_k(G) = k(n - \alpha)$. Instead of $\sigma_k(G)$, sometimes we simply write $\sigma_k$.

If $G$ contains a *hamiltonian cycle* (a cycle passing all vertices of $G$), then $G$ is called *hamiltonian*; otherwise, $G$ is *nonhamiltonian*. A graph $G$ with a *hamiltonian path* (a path passing all vertices of $G$) is said to be *traceable*. Let $C_k$ be the cycle of length $k$. The graph $G$ is said to be $k$-*connected* if $G - X$ is connected for any $X \subseteq V$ with $|X| < k < n$. Note that a tough graph is 2-connected, and toughness is a necessary condition for the existence of a hamiltonian cycle in a graph [6]. There is a polynomial algorithm $O(n^3)$ time to recognize 2-connected graph.

The problem $HP, HC$ are well-known $NPC$-problem [1] [10].

**$HP$ (HAMILTONIAN PATH)**
*Instance*: Graph $G$.
*Question*: Is $G$ traceable?

**$HC$ (HAMILTONIAN CYCLE)**
*Instance*: Graph $G$.
*Question*: Is $G$ hamiltonian?

A lot of authors have been studied Hamiltonian Cycles in graphs with large degree sums $\sigma_k$, but only for $k = 1, 2, 3$, (see [3] [5] [9], etc).

For a positive integer $k$, we state the problem $HCk$ as follow:

**$HCk$**
*Instance*: Given a real $t > 0$ and a graph $G$ satisfying $\sigma_k \geq \frac{kn}{2} t$.
*Question*: Is $G$ hamiltonian?

In [7], [8], we prove that:

**Theorem 1.1 [7].** *$HC2(t < 1)$ is $NPC$ and $HC2(t \geq 1)$ is $P$.*

**Theorem 1.2 [8].** *$HC3(t < 1)$ is $NPC$ and $HC3(t \geq 1)$ is $P$.*

In this paper, we study the class of graphs satisfying $\sigma_4 \geq 2n$ for the problem $HC4$.

## 2. RESULTS

The following Theorem will be proved in Section 5.

**Theorem 2.1**. *Let $G$ be 2-connected graph with $\sigma_4 \geq 2n$. If $G$ is nonhamiltonian then $\alpha(G) = 3$ and $G$ belongs to one of the following three classes of graphs:*

1. *Class $\mathcal{F}_1$ of 2-connected graphs $G$ with $\alpha(G) = 3$ such that there exists a subset $S \subseteq V(G), |S| = 2$ so that $G - S = K_{n_1} \cup K_{n_2} \cup K_{n_3}$.*

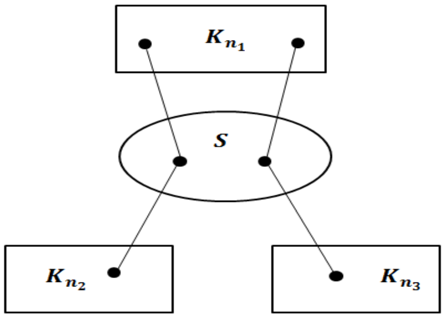


**Figure 1. Class $\mathcal{F}_1$.**

2. *Class $\mathcal{F}_2$ of 2-connected graphs $G$ with $\alpha(G) = 3$ such that there exists three disjoint complete graphs $K_{n_1}, K_{n_2}, K_{n_3} \subseteq G$ and a vertex $x \in V(G)$ and $y_1 \in K_{n_1}$, $y_2 \in K_{n_2}$, $y_3 \in K_{n_3}$ so that $G - \{x\} = (K_{n_1} \cup K_{n_2} \cup K_{n_3}) + \{y_1y_2, y_2y_3, y_3y_1\}$. Moreover, there exists three vertices $z_1 \in K_{n_1} - \{y_1\}$, $z_2 \in K_{n_2} - \{y_2\}$, $z_3 \in K_{n_3} - \{y_3\}$ such that $z_1, z_2, z_3 \in N(x)$ and $x$ can possibly be adjacent to the another vertices.*

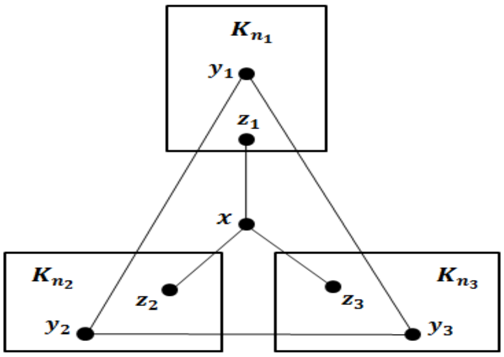


**Figure 2. Class $\mathcal{F}_2$.**

3. *Class $\mathcal{F}_3$ of 2-connected graphs $G$ with $\alpha(G) = 3$ such that there exists three disjoint complete graphs $K_{n_1}, K_{n_2}, K_{n_3} \subseteq G$ $(|K_{n_1}|, |K_{n_2}|, |K_{n_3}| \geq 3)$ and distinct vertices $y_i, z_i \in K_{n_i}$ for $i = 1, 2, 3$ so that $G = K_{n_1} \cup K_{n_2} \cup K_{n_3} + \{y_1y_2, y_2y_3, y_3y_1\} + \{z_1z_2, z_2z_3, z_3z_1\}$.*

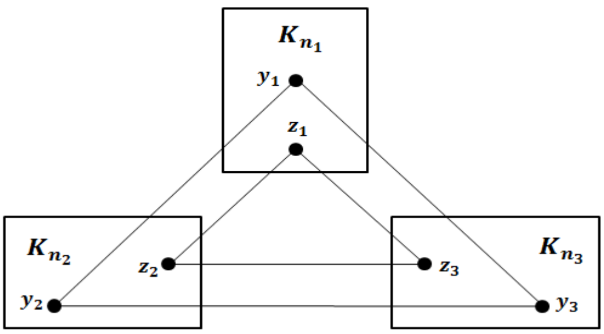


**Figure 3. Class $\mathcal{F}_3$.**

Note that the graph $G_n = K_1 * K_1 * (\overline{K}_3 * K_{n-5})$ with $n \geq 11$ satisfies $\sigma_4 \geq 2n$ and is not 2-connected. In Section 3, we give polynomial algorithms to recognize whether a given graph belongs to $\mathcal{F}_1 \cup \mathcal{F}_2 \cup \mathcal{F}_3$.

From Theorem 2.1, we conclude the following corollary.

**Corollary 2.1**. *Every 2-connected graph with $\alpha \geq 4$ and $\sigma_4 \geq 2n$ is hamiltonian.*

For $t < 1$, we prove the following Theorem:

**Theorem 2.2.** *$HC4$ $(t < 1)$ is $NPC$.*

*Proof*. The $HC4$ is a subproblem of $HC$, so it belongs to $NP$. In order to prove $HC4$ $(t < 1)$ is $NPC$, we will construct a polynomial transformation from the problem $HP$ to it.

For any graph $G_1$ with $n_1$ vertices, we choose a positive integer $m \geq max\left\{\frac{t(n_1-1)}{2(1-t)}, 5\right\}$. Then we construct a graph $G_2$ from $G_1$ by adding new vertex set $\{p_1, p_2, \ldots, p_m\} \cup \{q_1, q_2, \ldots, q_{m-1}\}$ and the edges joining each vertex of $\{p_1, p_2, \ldots, p_m\}$ to all other vertices. In this way, we obtain the graph $G_2 = (G_1 \cup \overline{K}_{m-1}) * K_m$. This construction can be proceeded with the Turing machine in polynomial time.

We observe that the graph $G_2$ has $n_2 = n_1 + 2m - 1$ vertices and $\sigma_4(G_2) = 4m$. Because of $m \geq \frac{t(n_1-1)}{2(1-t)}$, so $2m \geq t(n_1 + 2m - 1)$, it implies that $\sigma_4(G_2) \geq 2n_2 t$.

Now we prove that $G_2$ has a hamiltonian cycle if and only if $G_1$ has a hamiltonian path. Indeed, if $G_1$ has a hamiltonian path $H$ then $C = (H, p_1, q_1, p_2, q_2, \ldots, p_{m-1}, q_{m-1}, p_m)$ is a hamiltonian cycle in $G_2$.

If $G_2$ has a hamiltonian cycle $C$. Observe that $q_i$ $(i = 1..m-1)$ has only neighbor $p_j$ $(j = 1..m)$, so all vertices in $\{q_1, q_2, \ldots, q_{m-1}\}$ are only adjacent to all the vertices in $\{p_1, p_2, \ldots, p_m\}$. Then, if we remove all vertices in $\{p_1, p_2, \ldots, p_m\}$ then we obtain $m$ connected components, which are $\{q_1\}, \{q_2\}, \ldots, \{q_{m-1}\}$ and $G_1$, each of the connected components has a hamiltonian path (the rest of $C$ after removing $\{p_1, p_2, \ldots, p_m\}$). Therefore, $G_1$ has a hamiltonian path.

Thus, we have a polynomial transformation from $HP$ to $HC4(t < 1)$. Since $HC4(t < 1) \in NP$ and $HP \in NPC$, it implies that $HC4$ $(t < 1) \in NPC$.

**Theorem 2.3.** *$HC4$ $(t \geq 1)$ is $P$.*

*Proof.* Assume that $G$ satisfies $\sigma_4 \geq 2nt$ with $t \geq 1$. First, we check whether $G$ is 2-connected or not (it can be done in polynomial time).

If $G$ is not 2-connected then $G$ is nonhamiltonian.

If $G$ is 2-connected, then by Theorem 2.1, either $G$ is hamiltonian or $G$ belongs to $\mathcal{F}_1 \cup \mathcal{F}_2 \cup \mathcal{F}_3$ which can be recognize in polynomial time (see Section 3). Thus, $HC4$ $(t \geq 1)$ is $P$.

## 3. POLYNOMIAL ALGORITHMS RECOGNIZING THE CLASSES $\mathcal{F}_1, \mathcal{F}_2, \mathcal{F}_3$

Assume that $S \subseteq V(G)$ and $H_1, H_2, \dots., H_k$ are connected components of $G - S$. Note that the problem "Given a vertex set $S$ in a graph $G$, determine $\omega(G - S)$ and whether every connected component of $G - S$ is complete" can be solved in polynomial time by an algorithm $O(n^2)$. Following, we design the polynomial algorithms recognizing the classes $\mathcal{F}_1, \mathcal{F}_2, \mathcal{F}_3$.

### 3.1. Algorithm recognizing the class $\mathcal{F}_1$

Every graph $G$ in class $\mathcal{F}_1$ is not 1-tough. If we remove $S$, then we get three connected components which are complete.

*Input:* graph $G$ with $\sigma_4 \geq 2n$.

*Output:* Is_Graph_$\mathcal{F}_1$ return True if $G \in \mathcal{F}_1$, else return False.

***Algorithm:***

**Function Boolean Is_Graph_$\mathcal{F}_1$**
**Begin**

```
If G is not 2-connected Then Return False;
For each S in V(G)^2 do
If (ω(G − S) = 3) and (the connected components
H1, H2, H3 are complete) Then Return True;
Return False;
```

**End**;

Checking $G$ is not 2-connected can be done by $O(n^2)$ time. Next, there are $C_n^2$ iterations, each iteration requires $O(n^2)$ time. Thus the overall time required by algorithm Is_Graph_$\mathcal{F}_1$ is $O(n^4)$.

### 3.2. Algorithm recognizing the class $\mathcal{F}_2$

For each graph $G$ in class $\mathcal{F}_2$, if we remove $S = \{x, y_1, y_2, y_3\}$, then we get three connected components $H_1, H_2, H_3$ which are complete.

*Input:* graph $G$ with $\sigma_4 \geq 2n$.

*Output:* Is_Graph_$\mathcal{F}_2$ return True if $G \in \mathcal{F}_2$, else return False.

***Algorithm:***

**Function Boolean Is_Graph_$\mathcal{F}_2$;**
**Begin**

```
    For each S in V(G)^4 do
    If (ω(G − S) = 3) and (the connected components
    H1, H2, H3 are complete) Then
      If there exists x ∈ S and S − {x} = {y1, y2, y3} such
      that:
        (|N_H1(x)|, |N_H2(x)|, |N_H3(x)| ≥ 1) and
        ({y1y2, y2y3, y3y1} ⊆ E(G)) and
        (H1 + {y1}, H2 + {y2}, H3 + {y3} are complete)
      Then Return True;
    Return False;
  End;
```

There are $C_n^4$ iterations, each iteration requires $O(n^2)$ time, so the overall time required by algorithm Is_Graph_$\mathcal{F}_2$ is $O(n^6)$.

### 3.3. Algorithm recognizing the class $\mathcal{F}_3$

For each graph $G$ in class $\mathcal{F}_3$, if we remove $S = \{y_1, y_2, y_3, z_1, z_2, z_3\}$, then we get three connected components $H_1, H_2, H_3$ which are complete.

*Input:* graph $G$ with $\sigma_4 \geq 2n$.
*Output:* Is_Graph_$\mathcal{F}_3$ return True if $G \in \mathcal{F}_3$, else return False.

***Algorithm:***

```
Function Boolean Is_Graph_F3
Begin
  For each S in V(G)^6 do
  If (ω(G − S) = 3) and (the connected components
  H1, H2, H3 are complete graphs) Then
    If there exists y1, y2, y3 ∈ S and S − {y1, y2, y3} =
    {z1, z2, z3} such that:
      ({y1y2, y2y3, y3y1, z1z2, z2z3, z3z1} ⊆ E(G)) and
      (H1 + {y1, z1}, H2 + {y2, z2}, H3 + {y3, z3} are complete)
    Then Return True;
  Return False;
End;
```

There are $C_n^6$ iterations, each iteration requires $O(n^2)$ time, so the overall time required by algorithm Is_Graph_$\mathcal{F}_3$ is $O(n^8)$.

## 4. PRELIMINARIES

For what follows we assume that $C$ is a longest cycle of $G$. On $\overrightarrow{C}$ ($C$ with a given orientation), we denote the predecessor and successor (along $\overrightarrow{C}$) by $x^-, x^+$, and $x^{++} = (x^+)^+, x^{--} = (x^-)^-$. In general, for a positive integer $i$, $x^{+i} = (x^{+(i-1)})^+$ and $x^{-i} = (x^{-(i-1)})^-$. Moreover, for a vertex set $A \subseteq V(C)$, we wirte $A^+ = \{x^+ : x \in A\}$ and $A^- = \{x^- : x \in A\}$. The path joining two vertices $x$ and $y$ of $C$, along $\overrightarrow{C}$, is denoted by $x\overrightarrow{C}y$, and the same path in reverse order are given by $y\overleftarrow{C}x$.

In this paper, we consider the paths and cycles as vertex sets. If $x, y$ are the end vertices of a path $P$, sometimes we write $xPy$ instead of $P$.

Assume that $H$ is a connected component of $G - C$ and $N_C(H)$ is the set of neighbors in $C$ of all vertices in $H$. A *edge sequence* is a path joining two vertices on $C$ and its inner vertices belong to $G - C - H$. In particular, an edge joining 2 non-consecutive vertices on $C$ is also a edge sequence.

**Lemma 4.1.** *Let $G$ be a 2-connected graph. If $G$ is nonhamiltonian and $H$ is a connected component of $G - C$ then*

(a) $N_C(H) \cap N_C(H)^+ = N_C(H) \cap N_C(H)^- = \emptyset$.

(b) *There is no edge sequence joining 2 vertices of $N_C(H)^+$. Similarly, there is no edge sequence joining 2 vertices of $N_C(H)^-$.*

(c) *If $v_i, v_j \in N_C(H)$ for $i \neq j$ then there is no vertex $z \in v_i^+\overrightarrow{C}v_j$ such that $\{v_i^+z^+, v_j^+z\} \subseteq E(G)$. Similarly, there is no vertex $z \in v_j^+\overrightarrow{C}v_i$ such that $\{v_j^+z^+, v_i^+z\} \subseteq E(G)$.*

(d) *For any $x \in H$ and for any $v_i \in N_C(H)$, $d(x) + d(v_i^+) \leq n - 1$.*

*Proof.* (a), (b), (c) are presented in [2], so we will prove (d). For any $x \in H$, $d(x) = |N_H(x)| + |N_C(x)| \leq |H| - 1 + |N_C(H)|$. By (a) and (b), $d(v_i^+) \leq (|G| - |H|) - |N_C(H)^+| = |G| - |H| - |N_C(H)|$, so $d(x) + d(v_i^+) \leq |H| - 1 + |N_C(H)| + |G| - |H| - |N_C(H)| = |G| - 1 = n - 1$.

**Lemma 4.2 [2].** *Assume that $u, v$ are nonadjacent vertices and $d(u) + d(v) \geq n$. Then $G$ is hamiltonian if and only if $G + uv$ is hamiltonian.*

We conclude the following Lemma from Lemma 4.2.

**Lemma 4.3.** *Assume that $G^* \supseteq G$ such that $V(G^*) = V(G)$ and $d_G(u) + d_G(v) \geq n$ for any edge $uv \in E(G^*) - E(G)$. Then $G$ is hamiltonian if and only if $G^*$is hamiltonian.*

## 5. PROOFS OF THEOREM 2.1

For what follows, we assume that $G$ is nonhamiltonian. Because $G$ is 2-connected, so $G$ is cycleable. Let $H_1, H_2, \dots, H_m$ be the connected components of $G - C$. Clearly, $|N_C(H_i)| \geq 2$ for any $i = 1..m$.

**Proposition 5.1.** *$H_1, H_2, \dots, H_m$ are complete graphs.*

*Proof.* We consider a connected component $H_t$ $(t = 1..m)$. Because $G$ is 2-connected, so $|N_C(H_t)| \geq 2$ and there are at least two vertices $v_i, v_j \in N_C(H_t)$. If $H_t$ is not complete then there are two distinct vertices $x, y \in H_t$ such that $xy \notin E(G)$. By Lemma 4.1 (a, b), $\{x, y, v_i^+, v_j^+\}$ is an independent vertex set, therefore by $\sigma_4 \geq 2n$, $d(x) + d(y) + d(v_i^+) + d(v_j^+) \geq 2n$. However, by Lemma 4.1 (d), $d(x) + d(v_i^+) \leq n - 1$ and $d(y) + d(v_j^+) \leq n - 1$, it implies that $d(x) + d(y) + d(v_i^+) + d(v_j^+) \leq 2n - 2$, a contradiction. Thus, $H_t$ is complete, and we have $H_1, H_2, \dots, H_m$ are complete graphs.

**Proposition 5.2.** $|N_C(H_t)| \leq \frac{|C|}{2}$ *for every* $t = 1..m$.

*Proof.* By Lemma 4.1 (a), $N_C(H_t) \cap N_C(H_t)^+ = \emptyset$, therefore $|C| \geq |N_C(H_t) \cup N_C(H_t)^+| = 2|N_C(H_t)|$, it implies that $|N_C(H_t)| \leq \frac{|C|}{2}$.

**Proposition 5.3.** $m = 1$.

*Proof.* We consider the case of $m$ as follow:

a) $m \geq 4$.

Let $x_i \in H_i$ for each $i = 1..4$. Clearly, the vertex set $\{x_1, x_2, x_3, x_4\}$ is independent, by $\sigma_4 \geq 2n$ we have $d(x_1) + d(x_2) + d(x_3) + d(x_4) \geq 2n$. Moreover, by Proposition 5.2 and $m \geq 4$, $d(x_i) \leq |H_i| - 1 + |N_C(H_i)| \leq |H_i| - 1 + \frac{|C|}{2}$, so $d(x_1) + d(x_2) + d(x_3) + d(x_4) \leq |H_1| + |H_2| + |H_3| + |H_4| + 2|C| - 4 \leq n + |C| - 4$, therefore $n + |C| - 4 \geq 2n$, it implies that $|C| \geq n + 4$, a contradiction.

Thus, the case $m \geq 4$ does not happen.

b) $m = 3$.

Let $x \in H_1, y \in H_2, z \in H_3$ and we consider each vertex $v_i \in N_C(H_1)$.

**Claim 5.1.** $v_i^+ \in N(y) \cup N(z)$.

*Proof.* Assume to the contrary that $v_i^+ \notin N(y) \cup N(z)$, then the vertex set $\{x, y, z, v_i^+\}$ is independent, by $\sigma_4 \geq 2n$ we have $d(x) + d(y) + d(z) + d(v_i^+) \geq 2n$. By Lemma 4.1 (d), $d(x) + d(v_i^+) \leq n - 1$.

Moreover, by Proposition 5.2 we have $d(y) \leq |H_2| - 1 + \frac{|C|}{2}$ and $d(z) \leq |H_3| - 1 + \frac{|C|}{2}$. Therefore, $d(x) + d(y) + d(z) + d(v_i^+) \leq n - 3 + |H_2| + |H_3| + |C| < 2n - 3$, a contradiction.

**Claim 5.2.** $|N_C(H_1)| = |N_C(H_2)| = |N_C(H_3)| = 2$.

*Proof.* If $|N_C(H_1)| \geq 3$ then by Claim 5.1, there are at least two vertices $v_i, v_j \in N_C(H_1)$ such that $v_i^+, v_j^+ \in N(y)$ or $v_i^+, v_j^+ \in N(z)$, therefore there exists an edge sequence joining $v_i^+, v_j^+$, which contradicts to Lemma 4.1 (b). Thus, $|N_C(H_1)| = 2$. Similarly, we have $|N_C(H_2)| = |N_C(H_3)| = 2$.

**Claim 5.3.** $5 \leq |C| \leq 6$.

*Proof.* If there exists $v \in C$ such that $v \notin N_C(H_1) \cup N_C(H_2) \cup N_C(H_3)$, then the vertex set $\{x, y, z, v\}$ is independent, by $\sigma_4 \geq 2n$ we have $d(x) + d(y) + d(z) + d(v) \geq 2n$. However, $d(x) \leq |H_1| - 1 + |N_C(H_1)| = |H_1| + 1$, $d(y) \leq |H_2| + 1$, $d(z) \leq |H_3| + 1$, $d(v) \leq |C| - 1$, so $d(x) + d(y) + d(z) + d(v) \leq |H_1| + |H_2| + |H_3| + |C| + 2 = n + 2$. It implies that $n + 2 \geq 2n$ and $n \leq 2$, a contradiction. Therefore, $N_C(H_1) \cup N_C(H_2) \cup N_C(H_3) = |C|$, and by Claim 5.2, $|C| \leq 6$. Moreover, by Lemma 4.1 (a), $|C| \geq 4$. If $|C| = 4$ then by Lemma 4.1 (a) and Claim 5.1, there exists an edge sequence joining two vertices in $N_C(H_1)^+$, which contradicts Lemma 4.1 (b). Thus, we have $5 \leq |C| \leq 6$.

If $|C| = 5$, so $C = (v_1, v_2, v_3, v_4, v_5)$. Without loss of generality, by Lemma 4.1 (a, b) and Claim 5.1, we assume that $v_1, v_3 \in N_C(H_1)$, $v_2 \in N_C(H_2)$, $v_4 \in N_C(H_3)$. Then, $v_5 \in N_C(H_2)$ and $N_C(H_2)^+ = \{v_1, v_3\}$. It implies that there exists an edge sequence joining two vertices in $N_C(H_2)^+$, which contradicts Lemma 4.1 (b). Therefore, by Claim 5.3, $|C| = 6$, so $C = (v_1, v_2, v_3, v_4, v_5, v_6)$ and by Claim 5.2, $N_C(H_1) \cap N_C(H_2) = N_C(H_2) \cap N_C(H_3) = N_C(H_1) \cap N_C(H_3) = \emptyset$. Without loss of generality, by Lemma 4.1 (a, b) and Claim 5.1, there are two possible case as follow:

(1) Case $v_1, v_3 \in N_C(H_1)$, $v_2 \in N_C(H_2)$, $v_4 \in N_C(H_3)$. Observe that $v_6 \in N_C(H_3)$ and $v_5 \in N_C(H_2)$. Let $W_1, W_2, W_3$ be the paths in $H_1, H_2, H_3$ joining the pair of vertices $(v_1, v_3), (v_2, v_5), (v_4, v_6)$ respectively. Then, we have $C' = (v_1 W_1 v_3 v_2 W_2 v_5 v_4 W_3 v_6 v_1)$ is longer than $C$, which contradicts the fact that $C$ is a longest cycle of $G$.

(2) Case $v_1, v_4 \in N_C(H_1)$, $v_2 \in N_C(H_2)$, $v_5 \in N_C(H_3)$. Observe that $v_6 \in N_C(H_2)$ and $v_3 \in N_C(H_3)$. Let $W_1, W_2, W_3$ be the paths in $H_1, H_2, H_3$ joining the pair of vertices $(v_1, v_4), (v_2, v_6), (v_3, v_5)$ respectively. The, we have $C' = (v_1 W_1 v_4, v_3 W_3 v_5, v_6 W_2 v_2, v_1)$ is longer than $C$, a contradiction.

Thus, the case $m = 3$ does not happen.

c) $m = 2$.

Without loss of generality, assume that $|H_1| + |N_C(H_1)| \geq |H_2| + |N_C(H_2)|$.

**Claim 5.4.** $|N_C(H_1)| = 2$.

*Proof*. By $|N_C(H_1)| \geq 2$, assume to contrary that $|N_C(H_1)| \geq 3$. Let $x \in H_1, y \in H_2$. By Lemma 4.1 (b) there exists two vertices $v_i^+, v_j^+ \in N_C(H_1)^+ - N_C(H_2)$. By Lemma 4.1 (a, b), the vertex set $\{x, y, v_i^+, v_j^+\}$ is independent, so $d(x) + d(y) + d(v_i^+) + d(v_j^+) \geq 2n$. By Lemma 4.1 (d), $d(x) + d(v_i^+) \leq n - 1$. Moreover, $d(y) \leq |H_2| - 1 + |N_C(H_2)| \leq |H_1| - 1 + |N_C(H_1)|$, $d(v_j^+) \leq n - |H_1| - |N_C(H_1)|$. Therefore $d(x) + d(y) + d(v_i^+) + d(v_j^+) \leq 2n - 2$, a contradiction. Thus, $|N_C(H_1)| = 2$.

**Claim 5.5.** $|N_C(H_2)| = 2$.

*Proof*. Assume to contrary that $|N_C(H_2)| \geq 3$. Arguing similarly the proof of Claim 5.4, there exists two vertices $v_i^+, v_j^+ \in N_C(H_2)^+ - N_C(H_1)$. Let $x \in H_1, y \in H_2$. By Lemma 4.1 (a, b), the vertex set $\{x, y, v_i^+, v_j^+\}$ is independent, so $d(x) + d(y) + d(v_i^+) + d(v_j^+) \geq 2n$. By Lemma 4.1 (b) and by $v_i^+ \notin N(H_1) \cup N(H_2)$, $d(v_i^+) \leq |C| - |N_C(H_2)|$. Moreover, $d(x) \leq |H_1| - 1 + |N_C(H_1)| = |H_1| + 1$, $d(y) \leq |H_2| - 1 + |N_C(H_2)|$, $d(v_j^+) \leq |C| - |N_C(H_2)| \leq |C| - 2 \leq n - 4$. Therefore, $d(x) + d(y) + d(v_i^+) + d(v_j^+) \leq |H_1| + |H_2| + |C| + n - 4 = 2n - 4$, a contradiction. Thus, $|N_C(H_2)| = 2$.

By arguing similarly above, observe that $|N_C(H_1)^+ \cap N_C(H_2)| = 1 = |N_C(H_2)^+ \cap N_C(H_1)|$. Without loss of generality, we assume that $N_C(H_2) = \{v_i, v_i^{+2}\}$, $N_C(H_1) = \{v_j, v_i^+\}$ with $v_j \neq v_i^{+3}$ and $v_j^+ \neq v_i$. Because $G$ is 2-connected and $C$ is a longest cycle of $G$, so $|H_2| = 1$, i.e $H_2 = \{y\}$. Let $x \in H_1$ and $W_1$ be the path in $H_1$ joining $v_i^+$ to $v_j$.

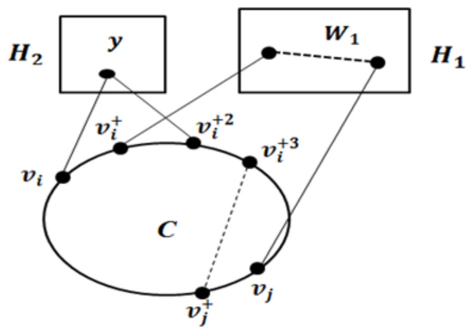


**Figure 4. Illustrating the proofs of part c), Proposition 5.3.**

If $v_i^{+3}v_j^+ \in E(G)$ then $C' = (v_i y v_i^{+2} v_i^+ W_1 v_j \overleftarrow{C} v_i^{+3} v_j^+ \overrightarrow{C} v_i)$ is longer than $C$, a contradiction. Therefore, $v_i^{+3}v_j^+ \notin E(G)$ and the vertex set $\{x, y, v_i^{+3}, v_j^+\}$ is independent, so $d(x) + d(y) + d(v_i^{+3}) + d(v_j^+) \geq 2n$. However, by Lemma 4.1 (d), $d(y) + d(v_i^{+3}) \leq n - 1$ and $(x) + d(v_j^+) \leq n - 1$, it implies that $d(x) + d(y) + d(v_i^{+3}) + d(v_j^+) \leq 2n - 2$, a contradiction.

Thus, the case $m = 2$ does not happen.

By these case a), b), c) do not happen, we finish the proof that $m = 1$. Then $G - C$ has only one connected component. For what follows, let $H$ be the connected component of $G - C$. The fact that $H = G - C$. By Proposition 5.1, $H$ is complete.

**Proposition 5.4.** $|N_C(H)| = 2$.

*Proof.* Clearly, $|N_C(H)| \geq 2$ by $G$ is 2-connected. Assume that $|N_C(H)| \geq 3$. For any two vertices $v_i, v_j \in N_C(H)$, let $v_k \in N_C(H) - \{v_i, v_j\}$ and $x \in H$, then by Lemma 4.1 (b) the vertex set $\{x, v_i^+, v_j^+, v_k^+\}$ is independent. So $d(x) + d(v_i^+) + d(v_j^+) + d(v_k^+) \geq 2n$. However, by Lemma 4.1 (d), $(x) + d(v_k^+) \leq n - 1$, it implies that $d(v_i^+) + d(v_j^+) \geq n + 1$.
By $G$ is 2-connected and $H$ is complete, there exists two vertices $v_{i_0}, v_{j_0} \in N_C(H)$ and a hamiltonian path $W$ in $H$ joining $v_{i_0}$ to $v_{j_0}$. Then $C' = (v_{i_0} W v_{j_0} \overleftarrow{C} v_{i_0}^+ v_{j_0}^+ \overrightarrow{C} v_{i_0})$ is a hamiltonian cycle of graph $G' = G + v_{i_0}^+ v_{j_0}^+$, i.e. $G'$ is hamiltonian. By Lemma 4.2, $G$ is hamiltonian if and only if $G'$ is hamiltonian, therefore $G$ is hamiltonian, which contradicts to the assumption that $G$ is nonhamiltonian. Thus, $|N_C(H)| = 2$.

For what follows, let $v_i, v_j$ be two vertices of $N_C(H)$ and let $W$ be the hamiltonian path of $H$ joining $v_i, v_j$.

**Proposition 5.5.** $N(v_i^+) \cup N(v_j^+) = C - \{v_i^+, v_j^+\}$.

*Proof.* Assume to the contrary that there exists $v_k \in C - \{v_i^+, v_j^+\}$ such that $v_k \notin N(v_i^+) \cup N(v_j^+)$. Clearly, $v_k \notin \{v_i, v_j\}$. Let $x \in H$, then by Lemma 4.1 (b), the vertex set $\{x, v_i^+, v_j^+, v_k\}$ is independent, so $d(x) + d(v_i^+) + d(v_j^+) + d(v_k) \geq 2n$. However, $d(x) \leq |H| + 1, d(v_k) \leq |C| - 3$, it implies that $d(v_i^+) + d(v_j^+) \geq 2n - |H| - |C| + 2 = n + 2$. Therefore, by Lemma 4.2, $G$ is hamiltonian if and only if $G' = G + v_i^+ v_j^+$ is hamiltonian. Observe that $C' = (v_i W v_j \overleftarrow{C} v_i^+ v_j^+ \overrightarrow{C} v_i)$ is a hamiltonian cycle of $G'$, i.e $G'$ is hamiltonian, it implies that $G$ is hamiltonian, a contradiction.

Now we consider two case of toughness of $G$.

## I. $G$ is not 1-tough

By $G$ is not 1-tough, there exists a vertex set $S \neq \emptyset$ such that $G - S$ has at least $|S| + 1$ connected components. By $G$ is 2-connected, $|S| \geq 2$. Since $n - |S| \geq \omega(G - S) \geq |S| + 1$ so $2|S| \leq n - 1$.

**Claim 5.6.** $S \cap H = \emptyset$.

*Proof.* Observe that $G - H = C$ is 1-tough, if $H - S = \emptyset$ then $\omega(G - S) = \omega(C - S) \leq |S|$, which contradicts to the fact that $\omega(G - S) \geq |S| + 1$. Therefore, $H - S \neq \emptyset$. Let $S \cap H = S_H, S \cap C = S_C$. If $|S_H| \geq 1$ then $\omega(G - S) \leq 1 + \omega(C - S_C) \leq 1 + |S_C| \leq |S|$, a contradiction. Thus, $|S_H| = 0$, i.e $S \cap H = \emptyset$.

Observe that $v_i, v_j \in S$, otherwise $\omega(G - S) \leq \omega(C - S) \leq |S|$, a contradiction. Therefore, $H$ is a connected component of $G - S$. Let $H, T_1, T_2, \ldots, T_k$ $(k \geq |S|)$ be the connected components of $G - S$.

**Claim 5.7.** $k = |S| = 2$

*Proof.* Assume that $k \geq 3$. Let $x \in H, y_1 \in T_1, y_2 \in T_2, y_3 \in T_3$, then the vertex set $\{x, y_1, y_2, y_3\}$ is independent, so $d(x) + d(y_1) + d(y_2) + d(y_3) \geq 2n$. Observe that $d(x) \leq |H| + 1$ and $d(y_i) \leq |T_i| - 1 + |S|$ for any $i = 1, 2, 3$. Therefore, $d(x) + d(y_1) + d(y_2) + d(y_3) \leq |H| + |T_1| + |T_2| + |T_3| + 3|S| - 2 \leq 2|S| - 2 + (n - k + 3) = 2|S| + n - k + 1$. It implies that $2|S| + n - k + 1 \geq 2n$, i.e $2|S| \geq n + k - 1 \geq n + 2$ (by $k \geq 3$), which contradicts to the fact that $2|S| \leq n - 1$. Therefore $k \leq 2$. By $k \geq |S| \geq 2$, we have $k = |S| = 2$.

By Claim 5.7 and by $v_i, v_j \in S$ we have $S = \{v_i, v_j\}$ and $G - S$ has three connected components, such as $H, T_1, T_2$. By Proposition 5.5, $T_1 = (\{v_i^+\} \cup N(v_i^+) - \{v_i, v_j\}$ and $T_2 = (\{v_j^+\} \cup N(v_j^+) - \{v_i, v_j\}$.

**Claim 5.8.** $T_1, T_2$ *is complete.*

*Proof.* Assume that $T_1$ is not complete. Then there exists pair of nonadjacent vertices $y, z \in T_1$. Let $x \in H$, then the vertex set $\{x, y, z, v_j^+\}$ is independent, so $d(x) + d(y) + d(z) + d(v_j^+) \geq 2n$. However, $d(x) \leq |H| + 1$, $d(v_j^+) \leq |T_2| + 1$, $d(y) \leq |T_1|$, $d(z) \leq |T_1|$. Therefore $d(x) + d(y) + d(z) + d(v_j^+) \leq |H| + 2|T_1| + |T_2| + 2 = n + |T_1|$. It implies that $n + |T_1| \geq 2n$, i.e $|T_1| \geq n$, a contradiction. Thus, $T_1$ is complete. Similarly, we have $T_2$ is complete.

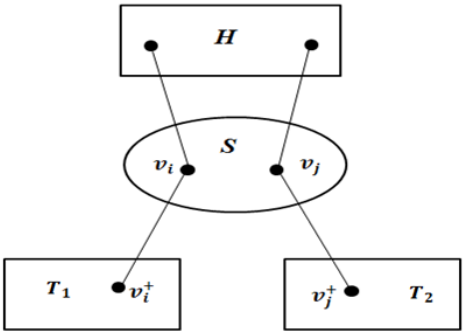


**Figure 5. Graph $G$ belongs to class $\mathcal{F}_1$.**

Clearly, $3 \leq \alpha(G) \leq 5$. If $\alpha(G) \geq 4$, there exists a independent set of four vertices, whose elements are $x \in H$, $y \in T_1$, $z \in T_2$ and a vertex in S (without loss of generality, assume that the vertex in $S$ is $v_i$). By $\sigma_4 \geq 2n$, we have $d(x) + d(y) + d(z) + d(v_i) \geq 2n$. However, $d(x) \leq |H| + 1$, $d(v_i) \leq n - 4$, $d(y) \leq |T_1|$, $d(z) \leq |T_2|$. It implies that $d(x) + d(y) + d(z) + d(v_i) \leq |H| + |T_1| + |T_2| + n - 3 = 2n - 5$, a contradiction. Thus, $\alpha(G) = 3$.

Conclude that in this Case $G$ is not 1-tough, $G$ belongs to class $\mathcal{F}_1$.

## II. $G$ is 1-tough

Let $P_1 = N(v_i^+) \cup \{v_i^+\}$, $P_2 = N(v_j^+) \cup \{v_j^+\}$. By Lemma 4.1 (c) and Proposition 5.5, we have $P_1, P_2$ are two paths on $C$ satisfying $\{v_i, v_i^+, v_i^{+2}\} \subseteq P_1$, $\{v_j, v_j^+, v_j^{+2}\} \subseteq P_2$, $P_1 \cup P_2 = C$ and if $v \in P_1 \cap P_2$ then $v$ is an end vertex of both $P_1, P_2$.

Let $A_1 = P_1 - \{v_i\}$, $A_2 = P_2 - \{v_j\}$. Clearly, $|A_1 \cap A_2| \leq 2$. We consider three case of $|A_1 \cap A_2|$.

### Case 1. $A_1 \cap A_2 = \emptyset$.

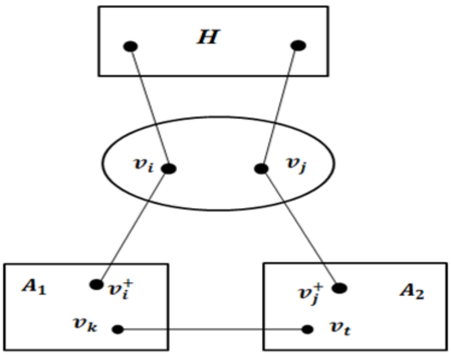


**Figure 6. Illustrating the Case 1.**

Observe that there exists an edge joining a vertex $v_k \in A_1$ to a vertex $v_t \in A_2$, otherwise $\omega(G - \{v_i, v_j\}) = 3$, which contradicts to the fact that $G$ is 1-tough.

If there exists pair of nonadjacent vertices $v_{i_1}, v_{i_2} \in A_1$, let $x \in H$, then the vertex set $\{x, v_{i_1}, v_{i_2}, v_j^+\}$ is independent, so $d(x) + d(v_{i_1}) + d(v_{i_2}) + d(v_j^+) \geq 2n$. By Lemma 4.1 (d), $d(x) + d(v_j^+) \leq n - 1$, we have $d(v_{i_1}) + d(v_{i_2}) \geq n + 1$. By Lemma 4.2, $G$ is hamiltonian if and only if $G' = G + v_{i_1}v_{i_2}$ is hamiltonian.

Arguing similarly, for any pair of nonadjacent vertices $v_{j_1}, v_{j_2} \in A_2$, we have $d(v_{j_1}) + d(v_{j_2}) \geq n + 1$ and $G$ is hamiltonian if and only if $G'' = G + v_{j_1}v_{j_2}$ is hamiltonian.

Let $G^*$ be the graph obtain from $G$ by adding new edges joining all pair of nonadjacent vertices in the same set $A_1$, respectively in $A_2$. By Lemma 4.3, $G$ is hamiltonian if and only if $G^*$ is hamiltonian. We consider graph $G^*$, let $W_1$ be the hamiltonian path of $A_1$ joining $v_i^+$ to $v_k$, and let $W_2$ be the hamiltonian path of $A_2$ joining $v_t$ to $v_j^+$. Then, we have $C' = (v_i v_i^+ W_1 v_k v_t W_2 v_j^+ v_j W v_i)$ is a hamiltonian cycle in $G^*$, i.e $G^*$ is hamiltonian. Therefore, $G$ is hamiltonian, a contradiction.

Thus, the Case 1 does not happen.

**Case 2.** $|A_1 \cap A_2| = 1$.

Let $A_1 \cap A_2 = \{v_k\}$. Without loss of generality, assume that $v_k \in v_i^{+2}\overrightarrow{C}v_j^-$ $(v_i^{+2} \neq v_j)$.

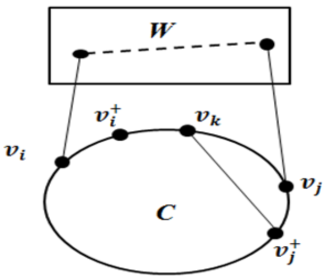


**Figure 7. Illustrating the Case 2.**

**Case 2.1.** $v_k \equiv v_i^{+2}$.

If $|H| > 1$ then $C' = (v_i W v_j \overleftarrow{C} v_k v_j^+ \overrightarrow{C} v_i)$ is longer than $C$. Therefore, $|H| = 1$, let $H = \{x\}$. If $v_i^- \in N(v_i^+)$ then $C' = (v_i x v_j \overleftarrow{C} v_k v_j^+ \overrightarrow{C} v_i^- v_i^+ v_i)$ is a hamiltonian cycle in $G$ a contradiction. Therefore, $v_i^- \notin N(v_i^+)$, and by Proposition 5.5, $v_i^- \in N(v_j^+)$ and $d(v_i^+) = 2$.

We consider subgraph $B_2 = A_2 - \{v_k\} = v_k\overrightarrow{C}v_i^- - \{v_k, v_j\}$. If there exists pair of nonadjacent vertices $v_{t_1}, v_{t_2} \in B_2$ then the vertex set $\{x, v_i^+, v_{t_1}, v_{t_2}\}$ is independent, so $d(x) + d(v_i^+) + d(v_{t_1}) + d(v_{t_2}) \geq 2n$. However, $d(x) = d(v_i^+) = 2$ and $d(v_{t_1}), d(v_{t_2}) \leq |C| - 3 = n - 4$, therefore $d(x) + d(v_i^+) + d(v_{t_1}) + d(v_{t_2}) \leq 2n - 4$, a contradiction. Thus, $B_2$ is complete.

If $v_j^- \neq v_k$ then $v_i^-, v_j^- \in B_2$, so $v_i^- v_j^- \in E(G)$, which contradicts to Lemma 4.1 (b). Therefore $v_j^- \equiv v_k$.

Because $G$ is 1-tough, nonhamiltonian, so $n \geq 7$ and $v_i^- \neq v_j^+$. If there exists a vertex $v_t \in v_j^{+2}\overrightarrow{C}v_i^-$ is adjacent to $v_j$ then we have $C' = (v_i x v_j v_t \overrightarrow{C} v_i^- v_t^- \overleftarrow{C} v_j^+ v_k v_i^+ v_i)$ is a hamiltonian cycle in $G$, a contradiction. Therefore, $v_j$ is not adjacent to all vertices in $v_j^{+2}\overrightarrow{C}v_i^-$.

Similarly, if there exists a vertex $v_t \in v_j^{+2}\overrightarrow{C}v_i^-$ is adjacent to $v_k$ then we have $C' = (v_i x v_j \overrightarrow{C} v_t^- v_i^- \overleftarrow{C} v_t v_k v_i^+ v_i)$ is a hamiltonian cycle in $G$, a contradiction. Therefore, $v_k$ is not adjacent to all vertices in $v_j^{+2}\overrightarrow{C}v_i^-$.

Conclude that the graph $G$ is shown in Figure 8, $v_i$ can possibly be adjacent to another vertices:

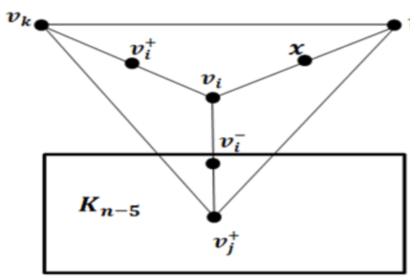


**Figure 8. Graph $G$ belongs to class $\mathcal{F}_2$.**

Clearly, $\alpha(G) = 3$ and $G$ belongs to class $\mathcal{F}_2$.

**Case 2.2.** $v_k \neq v_i^{+2}$ *and* $v_k \equiv v_j^-$.

Clearly, $v_k^- \neq v_i^+$. If $v_i^- v_k^- \in E(G)$, then $C' = (v_i W v_j v_k v_j^+ \overrightarrow{C} v_i^- v_k^- \overleftarrow{C} v_i)$ is a hamiltonian cycle of $G$, a contradiction. Therefore, $v_i^- v_k^- \notin E(G)$. By $v_k \equiv v_j^-$ and by Lemma 4.1 (b), $v_i^- v_k \notin E(G)$, so $v_i^- \neq v_j^+$ by $v_k \in N(v_j^+)$. We have the following Claims.

**Claim 5.9.** $v_i^- \in A_2 - A_1$.

*Proof.* Assume that $v_i^- \in A_1$. Let $x \in H$, then the vertex set $\{x, v_j^+, v_i^-, v_k^-\}$ is independent, so $d(x) + d(v_j^+) + d(v_i^-) + d(v_k^-) \geq$

$2n$. By Lemma 4.1 (d), $d(x) + d(v_j^+) \leq n - 1$ and $d(v_i^-) + d(v_k^-) \geq n + 1$. Therefore, by Lemma 4.2, $G$ is hamiltonian if and only if $G' = G + v_i^- v_k^-$ is hamiltonian. Observe that $C' = (v_i W v_j v_k v_j^+ \overrightarrow{C} v_i^- v_k^- \overleftarrow{C} v_i)$ is a hamiltonian cycle of $G'$, so $G'$ and $G$ are hamiltonian, a contradiction. Thus, $v_i^- \notin A_1$, and by $P_1 \cup P_2 = C$ we have $v_i^- \in A_2 - A_1$.

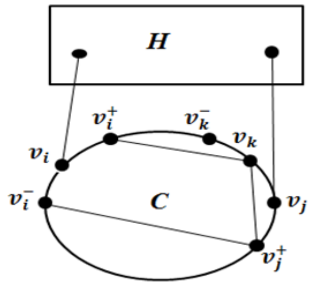


**Figure 9. Illustrating the Claim 5.9.**

Let $B_1 = A_1 - \{v_k\} = v_i^+ \overrightarrow{C} v_k^-$, $B_2 = A_2 - \{v_k\} = v_j^+ \overrightarrow{C} v_i^-$. By $v_i^- \neq v_j^+$ and by $v_k^- \neq v_i^+$ we have $|B_1|, |B_2| \geq 2$. Arguing similarly, for any pair of nonadjacent vertices $(y, z)$ in the same set $B_1$, respectively in $B_2$, we have $d(y) + d(z) \geq n + 1$.

**Claim 5.10.** *There are no edges joining a vertex in $B_1$ to a vertex in $B_2$.*

*Proof.* Assume to the contrary that there exists an edge joining $v_{t_1} \in B_1$ to $v_{t_2} \in B_2$. Clearly, $v_{t_1} \neq v_i^+, v_{t_2} \neq v_j^+$. Let $G^*$ be the graph obtain from $G$ by adding new edges joining all pair of nonadjacent vertices in the same set $B_1$, respectively in $B_2$. By Lemma 4.3, $G$ is hamiltonian if and only if $G^*$ is hamiltonian.

We consider graph $G^*$, observe that $B_1, B_2$ are complete. Let $W_1$ be the hamiltonian path in $B_1$ joining $v_{t_1}$ to $v_i^+$ and let $W_2$ be the path in $B_2$ joining $v_j^+$ to $v_{t_2}$. Then, $C' = (v_i W v_j v_k v_j^+ W_2 v_{t_2} v_{t_1} W_1 v_i^+ v_i)$ is a hamiltonian cycle of $G^*$, i.e $G^*$ is hamiltonian, it implies that $G$ is hamiltonian, a contradiction.

**Claim 5.11.** *$B_1, B_2$ are complete.*

*Proof.* Assume that there exists a pair of nonadjacent vertices $v_{i_1}, v_{i_2} \in B_1$. Let $x \in H$, then the vertex set $\{x, v_{i_1}, v_{i_2}, v_j^+\}$ is independent, so $d(x) + d(v_{i_1}) + d(v_{i_2}) + d(v_j^+) \geq 2n$. However, $d(x) \leq |H| + 1$, $d(v_j^+) \leq |B_2| + 2$ and $d(v_{i_1}), d(v_{i_2}) \leq |B_1| + 1$, therefore $d(x) + d(v_{i_1}) + d(v_{i_2}) + d(v_j^+) \leq |H| + 2|B_1| + |B_2| + 5 = n + |B_1| + 2$. It implies that $|B_1| \geq n - 2$, a contradiction. Thus, $B_1$ is complete. Similarly, we have $B_2$ is complete.

**Claim 5.12.** *$v_k, v_j$ are not adjacent to any vertex in $B_2 - \{v_j^+\}$.*

*Proof*. Assume that $v_k$ is adjacent to a vertex $v_p \in B_2 - \{v_j^+\}$. By Claim 5.11, let $W_1$ be the hamiltonian path of $B_1$ joining $v_k^-$ to $v_i^+$, and let $W_2$ be the hamiltonian path of $B_2$ joining $v_j^+$ to $v_p$. Then, $C' = (v_i W v_j v_j^+ W_2 v_p v_k v_k^- W_1 v_i^+ v_i)$ is a hamiltonian cycle of $G$, a contradiction. Similarly, if $v_j$ is adjacent to a vertex $v_q \in B_2 - \{v_j^+\}$, let $W_2^*$ be the hamiltonian path of $B_2$ joining $v_q$ to $v_j^+$, then $C' = (v_i W v_j v_q W_2^* v_j^+ v_k v_k^- W_1 v_i^+ v_i)$ is a hamiltonian cycle of $G$, a contradiction. Thus, $v_k, v_j$ are not adjacent to any vertex in $B_2 - \{v_j^+\}$.

**Claim 5.13.** *$v_j$ is not adjacent to any vertex in $B_1$.*

*Proof*. Assume to the contrary that $v_j$ is adjacent to a vertex $v_p \in B_1$. Let $W_2$ be the hamiltonian path of $B_2$ joining $v_j^+$ to $v_i^-$. It happens as one of two following case:

(1) Case $v_p \neq v_i^+$: Let $W_1$ be the hamiltonian path of $B_1$ joining $v_p$ to $v_i^+$, we have $C' = (v_i W v_j v_p W_1 v_i^+ v_k v_j^+ W_2 v_i^- v_i)$ is a hamiltonian cycle of $G$, a contradiction.

(2) Case $v_p \equiv v_i^+$: Let $W_1$ be the hamiltonian path of $B_1$ joining $v_p$ to $v_k^-$, we have $C' = (v_i W v_j v_p W_1 v_k^- v_k v_j^+ W_2 v_i^- v_i)$ is a hamiltonian cycle of $G$, a contradiction.

**Claim 5.14.** *$v_j$ is adjacent to all vertices in $H$.*

*Proof*. Assume to the contrary that $v_j$ is not adjacent to a vertex $x \in H$. Let $v_{t_1} \in B_1$, $v_{t_2} \in B_2 - \{v_j^+\}$, then by Claims 5.10, 5.12, 5.13, the vertex set $\{x, v_j, v_{t_1}, v_{t_2}\}$ is independent, so $d(x) + d(v_j) + d(v_{t_1}) + d(v_{t_2}) \geq 2n$. However, $d(x) \leq |H|$, $d(v_j) \leq |H| + 2$, $d(v_{t_1}) \leq |B_1| + 1$, $d(v_{t_2}) \leq |B_2|$. Therefore, $2n \leq d(x) + d(v_j) + d(v_{t_1}) + d(v_{t_2}) \leq 2|H| + |B_1| + |B_2| + 3 = n + |H|$, it implies that $|H| \geq n$, a contradiction.

**Claim 5.15.** *$v_k$ is adjacent to all vertices in $B_1$.*

*Proof*. Assume to the contrary that $v_k$ is not a vertex $v_{t_1} \in B_1$. Let $x \in H$ and $v_{t_2} \in B_2 - \{v_j^+\}$. Then by Claim 5.10 and by Claim 5.12, the vertex set $\{x, v_k, v_{t_1}, v_{t_2}\}$ is independent, so $d(x) + d(v_k) + d(v_{t_1}) + d(v_{t_2}) \geq 2n$. However, $d(x) \leq |H| + 1$, $d(v_k) \leq |B_1| + 2$, $d(v_{t_1}) \leq |B_1|$, $d(v_{t_2}) \leq |B_2|$. Therefore, $d(x) + d(v_k) + d(v_{t_1}) + d(v_{t_2}) \leq |H| + 2|B_1| + |B_2| + 3 = n + |B_1|$, it implies that $|B_1| \geq n$, a contradiction.

Let $H_1 = H + \{v_j\}$, by Claim 5.14, $H_1$ is complete. By Claim 5.15, $A_1 = B_1 + \{v_k\}$ is complete. The graph $G$ is shown in Figure 10, in which,

$H_1, A_1, B_2$ are complete and $|H_1|, |A_1|, |B_2| \geq 2$. Moreover, the vertex $v_i$ can possibly be adjacent to another vertices.

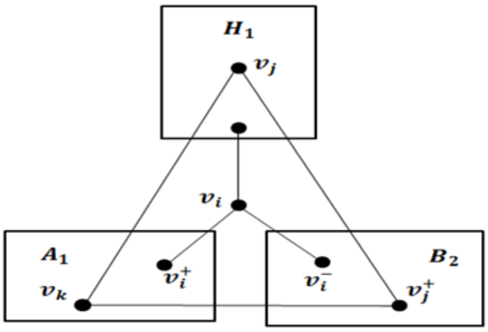


**Figure 10. Graph $G$ belongs to class $\mathcal{F}_2$.**

Clearly, $3 \leq \alpha(G) \leq 4$. If $\alpha(G) = 4$, then there exists $x \in H_1, y \in A_1, z \in B_2$ such that the vertex set $\{x, y, z, v_i\}$ is independent, so $d(x) + d(y) + d(z) + d(v_i) \geq 2n$. However, $d(x) + d(y) + d(z) \leq |H_1| + |A_1| + |A_2| - 1 = n - 2$, therefore $d(v_i) \geq n + 2$, a contradiction. Thus $\alpha(G) = 3$.

Conclude that in this Case 2.2, $G$ belongs to class $\mathcal{F}_2$.

**Case 2.3.** $v_k \neq v_i^{+2}$ *and* $v_k \neq v_j^-$.

Arguing similarly the proofs of Case 2.2, let $B_1 = A_1 - \{v_k\}$ and $B_2 = A_2 - \{v_k\}$, then for any pair of nonadjacent vertices $(y, z)$ together in $B_1$ or $B_2$, we have $d(y) + d(z) \geq n + 1$. Observe that $v_i^+ \neq v_k^- \in B_1$ and $v_j, v_j^+ \neq v_k^+ \in B_2$.

Let $G^*$ be the graph obtain from $G$ by adding new edges joining all pair of nonadjacent vertices in the same set $B_1$, respectively in $B_2$. By Lemma 4.3, $G$ is hamiltonian if and only if $G^*$ is hamiltonian. We consider graph $G^*$, let $W_1$ be the hamiltonian path of $B_1$ joining $v_k^-$ to $v_i^+$, and let $W_2$ be the hamiltonian path of $B_2$ joining $v_j^+$ to $v_k^+$. Then, we have $C' = (v_i W v_j v_j^+ W_2 v_k^+ v_k v_k^- W_1 v_i^+ v_i)$ is a hamiltonian cycle of $G^*$, i.e $G^*$ is hamiltonian. Therefore, $G$ is hamiltonian, a contradiction.

Thus, the Case 2.3 does not happen.

## Case 3. $|A_1 \cap A_2| = 2$.

Let $A_1 \cap A_2 = \{v_k, v_t\}$. Without loss of generality, we assume that $v_k \in v_i^{+2} \overrightarrow{C} v_j^-$ $(v_i^{+2} \neq v_j)$ and $v_t \in v_j^{+2} \overrightarrow{C} v_i^-$ $(v_j^{+2} \neq v_i)$. Let $B_1 = A_1 - \{v_k, v_t\}$, $B_2 = A_2 - \{v_k, v_t\}$. Arguing similarly the proofs of Case 2.2, for any pair of nonadjacent vertices $(y, z)$ in the same set $B_1$, respectively in $B_2$, we get $d(y) + d(z) \geq n + 1$.

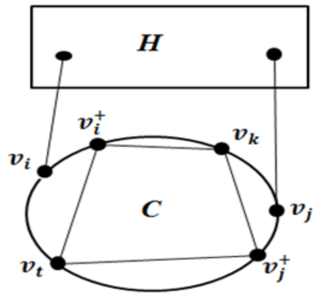


**Figure 11. Illustrating the Case 3.**

**Case 3.1.** $v_k \equiv v_i^{+2}$ *or* $v_t \equiv v_j^{+2}$.

Without loss of generality, assume that $v_k \equiv v_i^{+2}$. If $v_i^- \in N(v_i^+)$ then we have $C' = (v_i W v_j \overleftarrow{C} v_k v_j^+ \overrightarrow{C} v_i^- v_i^+ v_i)$ is a hamiltonian cycle of $G$, a contradiction. Therefore $v_i^- \notin N(v_i^+)$, i.e $v_i^- \notin A_1$ and $v_i^- \in A_2$. It implies that there is no vertex $v_t \in v_j^{+2} \overrightarrow{C} v_i^-$ such that $v_t \in A_1 \cap A_2$, a contradiction.

Thus, the Case 3.1 does not happen.

**Case 3.2.** $(v_k \neq v_i^{+2}$ *and* $v_k \equiv v_j^-)$ *or* $(v_t \neq v_j^{+2}$ *and* $v_t \equiv v_i^-)$.

Without loss of generality, assume that $v_k \neq v_i^{+2}$ and $v_k \equiv v_j^-$. We have the following Claims:

**Claim 5.16.** $v_t \equiv v_i^-$.

*Proof*. Assume to the contrary that $v_t \neq v_i^-$. Arguing similarly the proofs of Case 2.2, we have $v_i^- v_k^- \notin E(G)$ and $d(v_i^-) + d(v_k^-) \geq n+1$. By Lemma 4.2, $G$ is hamiltonian if and only if $G' = G + v_i^- v_k^-$ is hamiltonian. Observe that $C' = (v_i W v_j v_k v_j^+ \overrightarrow{C} v_i^- v_k^- \overleftarrow{C} v_i)$ is a hamiltonian cycle of $G'$, i.e $G'$ is hamiltoniania. It implies that $G$ is hamiltonian, a contradiction.

**Claim 5.17.** $|B_1|, |B_2| \geq 2$. *Moreover,* $v_i^{-2} \in B_2 - \{v_j^+\}$.

*Proof*. Because of $v_i^+, v_k^- \in B_1$, so $|B_1| \geq 2$. If $v_i^{-2} \equiv v_j^+$, then $C' = (v_i W v_j v_j^+ v_k \overleftarrow{C} v_i^+ v_i^- v_i)$ is a hamiltonian cycle of $G$, a contradiction. Therefore, $v_i^{-2} \neq v_j^+$. By Claim 5.16 we have $v_i^{-2} \in B_2 - \{v_j^+\}$ and $|B_2| \geq 2$.

**Claim 5.18.** *There are no edges joining a vertex in* $B_1$ *to a vertex in* $B_2$.

*Proof*. Assume to the contrary that there exists $v_{t_1} \in B_1$, $v_{t_2} \in B_2$ such that $v_{t_1} v_{t_2} \in E(G)$. Observe that $v_{t_1} \neq v_i^+$ and $v_{t_2} \neq v_j^+$. Let $G^*$ be the graph obtain from $G$ by adding new edges joining all pair of nonadjacent vertices in the same set $B_1$, respectively in $B_2$ (note that their degree sum is greater than $n+1$). By Lemma 4.3, $G$ is hamiltonian if and only if $G^*$ is hamiltonian. We consider the graph $G^*$, let $W_1$ be the hamiltonian path of $B_1$ joining $v_i^+$ to $v_{t_1}$, and let $W_2$ be the hamiltonian path of $B_2$ joining $v_{t_2}$ to $v_j^+$. Then $C' = (v_i W v_j v_k v_i^+ W_1 v_{t_1} v_{t_2} W_2 v_j^+ v_i^- v_i)$ is a hamiltonian cycle of $G^*$, i.e $G^*$ is hamiltonian. It implies that $G$ is hamiltonian, a contradiction.

**Claim 5.19.** *$B_1, B_2$ are complete.*

*Proof.* Assume that there exists a pair of nonadjacent vertices $v_p, v_q \in B_1$. Let $x \in H$, then the vertex set $\{x, v_p, v_q, v_j^+\}$ is independent, so $d(x) + d(v_p) + d(v_q) + d(v_j^+) \geq 2n$. However, $d(x) \leq |H| + 1$, $d(v_j^+) \leq |B_2| + 2$ and $d(v_p), d(v_q) \leq |B_1| + 2$. Therefore, $d(x) + d(v_p) + d(v_q) + d(v_j^+) \leq |H| + 2|B_1| + |B_2| + 7 = n + |B_1| + 3$. It implies that $|B_1| \geq n - 3$, a contradiction. Thus, $B_1$ is complete. Similarly, $B_2$ is complete.

**Claim 5.20.** *$v_i$ is not adjacent to all vertices in $B_1 - \{v_i^+\}$.*

*Proof.* Assume to the contrary that $v_i$ is adjacent to $v_{t_1} \in B_1 - \{v_i^+\}$. By Claim 5.19, let $W_1$ be the hamiltonian path of $B_1$ joining $v_i^+$ to $v_{t_1}$. We have $C' = (v_i W v_j v_k v_j^+ \overrightarrow{C} v_i^- v_i^+ W_1 v_{t_1} v_i)$ is a hamiltonian cycle of $G$, a contradiction.

**Claim 5.21.** *$v_i$ is not adjacent to all vertices in $B_2$.*

*Proof.* Assume to the contrary that $v_i$ is adjacent to $v_{t_2} \in B_2$. Observe that $v_{t_2} \neq v_j^+$, otherwise $C' = (v_i W v_j \overleftarrow{C} v_i^+ v_i^- \overleftarrow{C} v_j^+ v_i)$ is a hamiltonian cycle of $G$, a contradiction. Let $W_2$ be the hamiltonian path of $B_2$ joining $v_j^+$ to $v_{t_2}$. Then, $C' = (v_i W v_j v_k \overleftarrow{C} v_i^+ v_i^- v_j^+ W_2 v_{t_2} v_i)$ is a hamiltonian cycle of $G$, a contradiction.

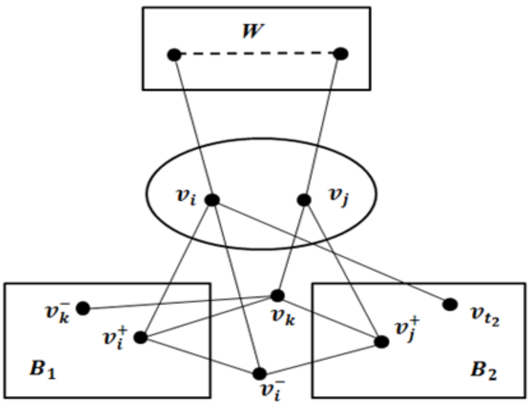


**Figure 12. Illustrating the proof of Claim 5.21.**

Similarly the proofs of Claim 5.20 and Claim 5.21, we have:

**Claim 5.22.** $v_j$ *is not adjacent to all vertices in* $B_1 \cup (B_2 - \{v_j^+\})$.

**Claim 5.23.** $v_i, v_j$ *are adjacent to all vertices in* $H$.

*Proof*. Assume that $v_i$ is not adjacent to $x \in H$. Let $v_p \in B_1 - \{v_i^+\}$, $v_q \in B_2 - \{v_j^+\}$. Then by Claims 5.18, 5.20, 5.21, the vertex set $\{x, v_i, v_p, v_q\}$ is independent, so $d(x) + d(v_i) + d(v_p) + d(v_q) \geq 2n$. However, $d(x) \leq |H|$, $d(v_i) \leq |H| + 3$, $d(v_p) \leq |B_1| + 1$, $d(v_q) \leq |B_2| + 1$, therefore $d(x) + d(v_i) + d(v_p) + d(v_q) \leq 2|H| + |B_1| + |B_2| + 5 = n + |H| + 1$. It implies that $|H| \geq n - 1$, a contradiction. Thus, $v_i$ is adjacent to all vertices in $H$. Similarly, $v_j$ is adjacent to all vertices in $H$.

**Claim 5.24.** $v_k$ *is not adjacent to all vertices in* $\{v_i\} \cup (B_2 - \{v_j^+\})$.

*Proof*. Assume that $v_k$ is adjacent to $v_{t_2} \in B_2 - \{v_j^+\}$. Let $W_1$ be the hamiltonian path of $B_1$ joining $v_k^-$ to $v_i^+$, and let $W_2$ be the hamiltonian path of $B_2$ joining $v_j^+$ to $v_{t_2}$. Then, we have $C' = (v_i W v_j v_j^+ W_2 v_{t_2} v_k v_k^- W_1 v_i^+ v_i^- v_i)$ is a hamiltonian cycle of $G$, a contradiction. Therefore, $v_k$ is not adjacent to all vertices in $B_2 - \{v_j^+\}$. Moreover, if $v_k$ is adjacent to $v_i$, by Claim 5.17, let $W_2'$ be the hamiltonian path of $B_2$ joining $v_j^+$ to $v_i^{-2}$. Then, $C' = (v_i W v_j v_j^+ W_2' v_i^{-2} v_i^- v_i^+ W_1 v_k^- v_k v_i)$ is a hamiltonian cycle of $G$, a contradiction. Thus, $v_k$ is not adjacent to $v_i$.

**Claim 5.25.** $v_k$ *is adjacent to all vertices in* $B_1$.

*Proof*. Assume to the contrary that $v_k$ is not adjacent to $v_{t_1} \in B_1$. Let $x \in H, v_{t_2} \in B_2 - \{v_j^+\}$, then by Claim 5.18 and by Claim 5.24, the vertex set $\{x, v_k, v_{t_1}, v_{t_2}\}$ is independent, so $d(x) + d(v_k) + d(v_{t_1}) + d(v_{t_2}) \geq 2n$. However, $d(x) \leq |H| + 1$, $d(v_k) \leq |B_1| + 2$, $d(v_{t_1}) \leq |B_1|$, $d(v_{t_2}) \leq |B_2|$. Therefore, $d(x) + d(v_k) + d(v_{t_1}) + d(v_{t_2}) \leq 2|B_1| + |B_2| + |H| + 3 = n + |B_1| - 1$. It implies that $|B_1| \geq n + 1$, a contradiction.

Arguing similarly the proofs of Claim 5.24 and Claim 5.25, we have:

**Claim 5.26.** $v_i^-$ *is not adjacent to all vertices in* $\{v_j\} \cup (B_1 - \{v_i^+\})$.

**Claim 5.27.** $v_i^-$ *is adjacent to all vertices in* $B_2$.

Observe that $v_k, v_i^- \in N_C(H)^-$, by Lemma 4.1 (b) we have:

**Claim 5.28.** $v_k v_i^- \notin E(G)$.

**Claim 5.29.** $v_i v_j \in E(G)$.

*Proof*. Assume to the contrary that $v_i v_j \notin E(G)$. Let $v_{t_1} \in B_1 - \{v_i^+\}$, $v_{t_2} \in B_2 - \{v_j^+\}$. Then by Claims 5.18, 5.20. 5.21 and 5.22, the vertex set $\{v_i, v_j, v_{t_1}, v_{t_2}\}$ is independent, so $d(v_i) + d(v_j) + d(v_{t_1}) + d(v_{t_2}) \geq 2n$. However, $d(v_i) \leq |H| + 2$, $d(v_j) \leq |H| + 2$, $d(v_{t_1}) \leq |B_1|$, $d(v_{t_2}) \leq |B_2|$. Therefore, $d(v_i) + d(v_j) + d(v_{t_1}) + d(v_{t_2}) \leq 2|H| + |B_1| + |B_2| + 4 = n + |H|$. It implies that $|H| \geq n$, a contradiction.

By Claim 5.25, $C_1 = B_1 + \{v_k\}$ is complete. By Claim 5.27, $C_2 = B_2 + \{v_i^-\}$ is complete. Moreover, by Claim 5.17, $|C_1|, |C_2| \geq 3$. By Claim 5.23 and Claim 5.29, $H_1 = H + \{v_i, v_j\}$ is complete and $|H_1| \geq 3$. Conclude that $G$ is shown in Figure 13, in which $H_1, C_1, C_2$ are complete and $|C_1|, |C_2|, |H_1| \geq 3$.

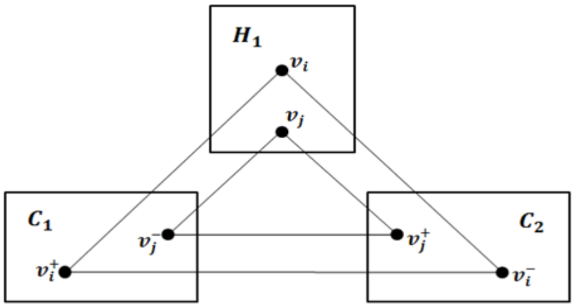


**Figure 13. Graph $G$ belongs to class $\mathcal{F}_3$.**

Clearly, $\alpha(G) = 3$ and $G$ belongs to class $\mathcal{F}_3$.

**Case 3.3.** $v_k \neq v_i^{+2}, v_k \neq v_j^-$ *and* $v_t \neq v_j^{+2}, v_t \neq v_i^-$.

Observe that $v_t^+, v_k^- \in B_1 - \{v_i^+\}$ and $v_t^-, v_k^+ \in B_2 - \{v_j^+\}$. Let $G^*$ be the graph obtain from $G$ by adding new edges joining all pair of nonadjacent vertices in the same set $B_1$, respectively in $B_2$ (note that their degree sum is greater than $n + 1$). By Lemma 4.3, $G$ is hamiltonian if and only if $G^*$ is hamiltonian.

We consider the graph $G^*$. Let $W_1$ be the hamiltonian path of $B_1$ joining $v_k^-$ to $v_i^+$, and let $W_2$ be the hamiltonian path of $B_2 - v_j^+$ joining $v_t^-$ to $v_k^+$. Then, we have $C' = (v_i W v_j v_j^+ v_t v_t^- W_2 v_k^+ v_k v_k^- W_1 v_i^+ v_i)$ is a hamiltonian cycle of $G^*$, i.e $G^*$ is hamiltonian, therefore $G$ is hamiltonian, a contradiction.

Thus, the Case 3.3 does not happen.

## 6. ACKNOWLEDGEMENTS

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